# 2 Bridging Unit: Polynomials and the binomial theorem

## Topic B: Algebraic division

Bridging to Ch2.3

When simplifying fractions, divide the numerator and denominator by their **highest common factor**.

For example, $\frac{112}{140}$ can be simplified to its equivalent fraction $\frac{4}{5}$ by dividing numerator and denominator by their HCF: 28, which is the product of their common prime factors 7, 2 and 2.

Algebraic fractions can be simplified in the same way. You must first factorise the numerator and the denominator, then divide both the numerator and denominator by their highest common factor.

For example, $\frac{x(x+1)(x-2)}{x^2(x+1)} = \frac{x-2}{x}$ since the common factor of $x(x+1)$ can be cancelled.

### Example 1

Simplify the fractions **a** $\frac{x^2+5x+6}{2x^2+6x}$ **b** $\frac{2x^2-3x-9}{4x^2-9}$

**a** $x^2+5x+6=(x+2)(x+3)$

$2x^2+6x=2x(x+3)$

Factorise numerator and denominator.

So $\frac{x^2+5x+6}{2x^2+6x}=\frac{(x+2)(x+3)}{2x(x+3)}=\frac{x+2}{2x}$

Cancel the common factor of $x+3$ from numerator and denominator.

**b** $2x^2-3x-9=2x^2-6x+3x-9$

Write $-3x$ as $-6x+3x$ since $3\times(-6)=-18$ and $2\times(-9)=-18$

$=2x(x-3)+3(x-3)$

Factorise in pairs.

$=(2x+3)(x-3)$

$4x^2-9=(2x+3)(2x-3)$

Recognise the denominator as a difference of two squares.

So $\frac{2x^2-3x-9}{4x^2-9}=\frac{(2x+3)(x-3)}{(2x+3)(2x-3)}=\frac{x-3}{2x-3}$

Cancel common factor $2x+3$ from numerator and denominator.

### Try It 1

Simplify the fractions **a** $\frac{x(x+2)(x-1)}{x^2(x-1)}$ **b** $\frac{3x^2+2x-1}{x^2-1}$

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If you can't simplify by cancelling common factors, you can use the method of **long division**. To use long division to divide the number 813 by 7

$$\begin{array}{r|lll} & 1 & & \\ \hline 7 & 8 & 1 & 3 \\ & 7 & \downarrow & \\ \cline{2-2} & 1 & 1 & \end{array}$$

- See how many times 7 goes into 8, ignoring remainders: only once, so write 1 in the hundreds column.
- Then multiply 7 by this result and write it under 813: $7 \times 1 = 7$
- Subtract this number from the 8 above it to get a remainder of 1, and write that underneath again. Then copy down the 1 in the tens column.

$$\begin{array}{r|lll} & 1 & 1 & \\ \hline 7 & 8 & 1 & 3 \\ & 7 & & \downarrow \\ \cline{2-2} & 1 & 1 & \\ & & 7 & \\ \cline{2-3} & & 4 & 3 \end{array}$$

- Repeat this process with the 11: again, 7 goes into 11 once, so write a 1, this time in the tens column.
- $7 \times 1 = 7$, so write a 7 underneath, and $11 - 7 = 4$, so write a 4 underneath that. Then copy down the 3
- Finally, 7 goes into 43 six times, so write a 6 in the units column, and $7 \times 6 = 42$, so write 42 underneath.
- $43 - 42 = 1$, so write 1 under that.
- To complete the solution, divide this 1 by the original divisor (7) to get your remainder: $813 \div 7 = 116\,\frac{1}{7}$

$$\begin{array}{r|lll} & 1 & 1 & 6 \\ \hline 7 & 8 & 1 & 3 \\ & 7 & & \\ \cline{2-2} & 1 & 1 & \\ & & 7 & \\ \cline{2-3} & & 4 & 3 \\ & & 4 & 2 \\ \cline{2-4} & & & 1 \end{array}$$

You can use long division in the same way to divide polynomials. For example, to work out $(6x^2 + x - 12) \div (2x + 3)$

$$\begin{array}{r|lll} & & 3x & -4 \\ \hline 2x+3 & 6x^2 & +\ x & -12 \\ & 6x^2 & +9x & \downarrow \\ \cline{2-3} & & -8x & -12 \\ & & -8x & -12 \\ \cline{3-4} & & & 0 \end{array}$$

- First divide $6x^2$ by $2x$ to give $3x$, and write this in the '$x$-column'.
- Multiply $3x$ by $2x + 3$ to get $6x^2 + 9x$ and write this underneath
- Subtract this from $6x^2 + x$ to get $-8x$ and write that underneath again, then carry down the $-12$
- $2x$ goes into $-8x$ negative four times, so write $-4$ in the next column. Then multiply this by $2x + 3$ to get $-8x - 12$ and write that underneath.
- Subtracting these two rows gives 0, so the answer is $3x - 4$ exactly (if there had been a remainder, you would need to divide it by $2x + 3$)

**Example 2**

Work out $(12x^2+7x+1)\div(3x-2)$

$$\begin{array}{r l}
 & 4x+5 \\
3x-2 \big) & 12x^2+7x+1 \\
 & \underline{12x^2-8x} \\
 & 15x+1 \\
 & \underline{15x-10} \\
 & 11
\end{array}$$

$$(12x^2+7x+1)\div(3x-2)=4x+5+\frac{11}{3x-2}$$

First divide $12x^2$ by $3x$ to give $4x$ and write above the $x$-column.

Multiply the divisor by $4x$ and subtract.

Copy down the 1

$15x \div 3x = 5$ so write this above the number column, then multiply by the divisor and subtract, giving a remainder of 11

Write the remainder over the divisor.

**Try It 2**

Use long division to work out $(5x^2+19x-1)\div(x+4)$

Another method of dividing polynomials involves **comparing the coefficients**. The **coefficient** is the number or constant that multiplies the variable. For example, for $2x^3-5x$, the coefficients of $x^3$ and $x$ are 2 and –5. You can equate coefficients of the same variable from both sides of an identity.

**Example 3**

Given that $(ax+1)(x+3)^2 \equiv 3x^3+bx^2+cx+9$, find the values of the constants $a$, $b$ and $c$

$$\begin{aligned}
(ax+1)(x+3)^2 &\equiv (ax+1)(x^2+6x+9) \\
&\equiv ax^3+6ax^2+9ax+x^2+6x+9 \\
&\equiv ax^3+(6a+1)x^2+(9a+6)x+9
\end{aligned}$$

So $3x^3+bx^2+cx+9 \equiv ax^3+(6a+1)x^2+(9a+6)x+9$

Expand the squared bracket.

Multiply by the first bracket.

Collect like terms.

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Equate coefficients of $x^3$: $3 - a$

Equate coefficients of $x^2$: $b = 6a + 1 = 19$

Equate coefficients of $x^3$: $c = 9a + 6 = 33$

Therefore, $a = 3$, $b = 19$ and $c = 33$

The coefficients of $x^3$ on both sides of the identity must be equal so $a = 3$

Using $a = 3$

**Try It 3**

Given that $(x+a)(x-2)(x+1) \equiv x^3 - 5x^2 + bx + c$, find the values of the constants $a$, $b$ and $c$

You can use the comparing coefficients technique and your knowledge of indices to divide algebraic expressions. As you saw in Example 2, dividing a quadratic polynomial by a linear polynomial gives a linear quotient and a constant remainder.

For example, for certain constants $a$, $b$ and $c$

$$\frac{15x^2 - 14x + 5}{3x - 4} \equiv ax + b + \frac{c}{3x-4}$$

**Dividend** — $15x^2 - 14x + 5$

**Divisor** — $3x - 4$

**Quotient** — $ax + b$

**Remainder** — $c$

This is an identity.

Use the general form for a linear expression.

To find the values of $a$, $b$ and $c$ you multiply both sides of the identity by $3x - 4$

$$15x^2 - 14x + 5 \equiv (ax+b)(3x-4) + c$$
$$\equiv 3ax^2 + (3b - 4a)x - 4b + c$$

Comparing coefficients of $x^2$ gives $3a = 15 \Rightarrow a = 5$

Comparing coefficients of $x$ gives $-14 = 3b - 4a \Rightarrow 3b = 4a - 14 = 6 \Rightarrow b = 2$

Comparing constant terms gives $5 = -4b + c \Rightarrow c = 5 + 4b = 13$

Therefore, $(15x^2 - 14x + 5) \div (3x - 4) \equiv 5x + 2 + \dfrac{13}{3x-4}$

The same method can be used for higher-order polynomials, for example, dividing a cubic polynomial by a linear polynomial will give a quadratic quotient and a constant remainder.


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**Key point**

Dividing a polynomial of order $m$ by a polynomial of order $n$ where $m \geq n$ will give

- A quotient of order $m - n$
- A remainder of order at most $n - 1$

For example, dividing two quadratics will give a constant quotient plus a linear remainder.

## Example 4

Use the method of comparing coefficients to work out $(2x^3 - 7x^2 - 19x + 23) \div (x - 5)$

$$\frac{2x^3 - 7x^2 - 19x + 23}{x - 5} \equiv ax^2 + bx + c + \frac{d}{x - 5}$$

A cubic (order 3) divided by a linear expression (order 1) gives a quotient of order 3 – 1 = 2, i.e. a quadratic, and a remainder of order 1 – 1 = 0, i.e. a constant. Write the general expression for a quadratic and a constant remainder.

$$2x^3 - 7x^2 - 19x + 23 \equiv (ax^2 + bx + c)(x - 5) + d$$

Multiply both sides by the divisor $x - 5$

$$\equiv ax^3 - 5ax^2 + bx^2 - 5bx + cx - 5c + d$$

$$\equiv ax^3 + (b - 5a)x^2 + (c - 5b)x - 5c + d$$

$x^3 : a = 2$

Compare coefficients of $x^3$, $x^2$, $x$ and constant terms to find the values of $a$, $b$, $c$ and $d$

$x^2 : b - 5a = -7 \quad \Rightarrow \quad b = 5a - 7 = 3$

$x : c - 5b = -19 \quad \Rightarrow \quad c = 5b - 19 = -4$

Constant $: -5c + d = 23 \quad \Rightarrow \quad d = 23 + 5c = 3$

Therefore $(2x^3 - 7x^2 - 19x + 23) \div (x - 5) = 2x^2 + 3x - 4 + \dfrac{3}{x - 5}$

Write out the solution.

## Try It 4

Use the method of comparing coefficients to work out $(6x^2 + 11x - 49) \div (3x - 5)$


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# Bridging Exercise Topic B

**1** Simplify these fractions.

**a** $\dfrac{x(x-5)(x+2)}{x^3(x+2)}$

**b** $\dfrac{(x+3)^2}{x(x+3)}$

**c** $\dfrac{(x-4)}{2x(x-4)}$

**d** $\dfrac{x^2(x+5)}{x(x+5)^2}$

**2** Simplify these fractions by first factorising the numerator and the denominator.

**a** $\dfrac{x^2-2x-8}{x^2+4x+4}$

**b** $\dfrac{x^2-10x+21}{x^2-x-6}$

**c** $\dfrac{x^2-3x-10}{x^2-10x+25}$


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**d** $\frac{x^2+10x+24}{2x+8}$

**e** $\frac{x^2+6x}{x^2-36}$

**f** $\frac{3x^2+6x}{x^2-5x-14}$

**g** $\frac{5x^3+15x^2}{x^2+6x+9}$

**h** $\frac{x^2-64}{3x^2-24x}$

**i** $\frac{25-x^2}{45-4x-x^2}$


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**j** $\dfrac{2x^2-x-28}{2x^3+7x^2}$

**k** $\dfrac{15x^2+7x-4}{10x^2+13x+4}$

**l** $\dfrac{x^3-100x}{6x^2+56x-40}$

**m** $\dfrac{12x^3+36x}{2x^2+6}$

**n** $\dfrac{42x^2-x-1}{36x^2-12x+1}$

**o** $\dfrac{9x^3-x}{24x^2-x-3}$


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**p** $\dfrac{9x^2-34x-8}{2x^4-8x^3}$

**3** Use long division to work out these expressions.

**a** $(2x^2-9x-16)\div(x-6)$

**b** $(6x^2+3x+2)\div(x-1)$



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**c** $(5x^2+41x+41)\div(x+7)$

**d** $(6x^2+x-2)\div(2x-1)$

**e** $(15x^2+26x+5)\div(3x+4)$


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**f** $(8x^2+6x-34)\div(4x+9)$

**g** $(3x^3+18x^2+9x+19)\div(x+5)$


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**h** $(12x^3+4x^2+13x+4)\div(2x+1)$

**4** Find the values of the constants $a$, $b$ and $c$ in each of these identities.

**a** $(x+3)(x-4)\equiv ax^2+bx+c$

**b** $(x-9)^2\equiv ax^2+bx+c$

**c** $(x+4)(x+2)(x-2)\equiv x^3+ax^2+bx+c$



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**d** $(x+5)^2(x+8) \equiv x^3 + ax^2 + bx + c$

**e** $(ax+1)(x+3)(x-7) \equiv 5x^3 + bx^2 + cx - 21$

**f** $(x+a)(x-1)(x+5) \equiv x^3 + bx^2 + cx + 10$


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**g** $9x^3 + bx^2 + cx - 2 \equiv (ax+1)^2(x-2)$ where $a > 0$

**h** $(x+a)(x+b)(x-4) \equiv x^3 - 8x^2 - 5x + c$ where $b > 0$

**5** Use the method of comparing coefficients to work out each of these divisions.

**a** $(3x^2 - 19x - 18) \div (x - 7)$



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**b** $(4x^2 + 27x + 20) \div (x + 6)$

**c** $(18x^2 + 3) \div (3x + 1)$


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**d** $(28x^2 - 120x - 34) \div (2x - 9)$

**6** Use the method of comparing coefficients to simplify each of these fractions.

**a** $\dfrac{3x^3 - 2x^2 - 3x + 6}{x - 1}$


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**b** $\dfrac{14x^3+27x^2+7x-4}{2x+3}$

**c** $\dfrac{3x^3+7x^2-11x+4}{x^2+3x-2}$

**d** $\dfrac{2x^3-x-1}{2x^2-4}$


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